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EXPERIMENTAL REPORT

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MECHANISM OF ARMOR PENETRATION

Fourth Partial Report

BY

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Senior Physicist

DATE 5 August 1944

WATERTOWN ARSENAL
WATERTOWN, MASS.

MECHANISM OF ARMOR PENETRATION

Fourth Partial Report

OBJECT

To lay a rational foundation for the mechanics of
armor penetration.

SUMMARY

A method is outlined for computing the stresses and
the deformation in armor during projectile impact. The
primary problem in this computation is the evaluation of
the effect of the plate material's inertia.

As a first approximation in the evaluation of the
effect of the plate's inertia, the force acting upon the
plate is assumed to be so distributed as to give rise to
no localized plastic deformation. Both the motion and
the bending moment of the plate may then be computed in
the region where the force is applied. While the normal
velocity of the plate is directly proportional to the
force, the bending moment, under conditions of combat,
increases continuously if the force is maintained constant.

As a second approximation in the evaluation of the
effect of the plate's inertia, the region of impact is
assumed to be so supported as to give the velocity of the
plate and the bending moments computed in the first ap-
proximation. The deformation is then assumed to be in the

nature of a static penetration under adiabatic conditions of plastic flow. The inertial resistance of the plate material associated with this plastic deformation may thereby be computed from the solution of the problem of a projectile being pushed slowly through a plate. At high velocities it is found that this inertial resistance has a high peak at the instant of impact, the peak being higher the blunter the projectile. This high initial inertial resistance may give rise to shatter of projectiles striking homogeneous armor. This peak may be smoothed out, and the tendency to shatter thereby lessened, by an A.P. cap.

When the impact of the projectile is too abrupt, which may arise either through a very high striking velocity or through a blunt ogive, the above approximation methods are no longer valid. In such cases the wave propagation of stress must be considered. Examples are given of the effect of such waves both in the plate and in the projectile. In the plate they may give rise to spalls being thrown off the back of the plate. In the case of the projectiles these waves may result in the detachment of the front of the ogive.

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INTRODUCTION

The foundations of the mechanics of armor penetration lie in the simultaneous solution of the equations of motion of the projectile material and of the plate material. The response of an A.P. projectile to a given force may be calculated, to a good approximation, by considering the projectile as an elastic body. This assumption allows the calculation of the general features of the stress inside a projectile during impact,[1] and from this knowledge a rational system for projectile design is being formulated which reduces to a minimum the likelihood of projectile deformation or fracture during penetration.[2] In this approximation the projectile is regarded essentially as a rigid body, and therefore its equations of motion are readily soluble. On the other hand, the equations of motion of the plate, which necessarily undergoes plastic deformation, are so complex that exact solutions are not to be expected. Nevertheless it is believed that a better understanding of the behavior of the plate may be obtained by finding, through

1. C. Zener and R. E. Peterson: "Principles of Projectile Design for Penetration, First Partial Report", Report Number WAL 762/231.
2. Series of reports on "Principles of Projectile Design for Penetration".
   C. Zener and J. Sullivan, Second Partial Report, Report Number WAL 762/231-2.
   D. Van Winkle, Third Partial Report, Report Number WAL 762/231-3.
   D. Van Winkle, Fourth Partial Report, Report Number WAL 762/231-4.

judicious assumptions, approximate solutions to the
fundamental equations rather than by guessing at the
plate's behavior based solely upon observations of the
plate after penetration has occurred. It is the purpose
of the present report to furnish a guide in the making
of such approximations.

The problem of the reaction of the plate to the pro-
jectile would be greatly simplified if attention could be
concentrated upon the plate material in the immediate vi-
cinity of the projectile. This simplification is ac-
complished by first solving the equations of motion of the
plate material outside the immediate vicinity of the pro-
jectile, i.e., outside the region where the material
undergoes plastic deformation, and then replacing this
material by an equivalent support.

If the velocity of the projectile is not too great,
the equations of motion of the plate material surrounding
the projectile may be simplified by neglecting the
inertial terms, i.e., the acceleration of the plate ma-
terial. The problem of plate response then reduces to the
comparatively simple problem of static penetration under
known conditions of support, the deformation being re-
garded as occurring adiabatically (no heat flow). These
inertial terms in the equations of motion are unfortu-
nately often not negligible under the conditions likely
to be encountered in combat. The effect of these inertial

terms is to increase the force with which the plate
reacts upon the projectile at the first instant of impact,
and it is this increase in force which is the cause for
projectile shatter.[1] When these inertial terms are not
negligible, it is possible, at not too high velocities,
to estimate their effects by a simplification which
regards the type of motion of the plate material as
independent of the incident velocity. In this case the
problem of the plate's response again reduces to the
problem of solving the equations for the plate material
under essentially static conditions, and finally using
the solutions so obtained to compute the effects of the
material's inertia. Under more severe conditions the
inertial terms have an appreciable effect upon the type
of deformation, i.e., the actual propagation of stress
must be considered. An attempt is made to describe the
reaction of the plate in such extreme cases, and typi-
cal consequences of stress waves are cited.

RESULTS AND DISCUSSION

I GROSS BEHAVIOR OF PLATE

Armor with no inertia would behave in a radically
different manner than does actual armor. No type of pro-
jectile could penetrate, or even scratch, a freely sus-
pended armor plate with no inertia. Such a plate would
acquire the velocity of the projectile at the first

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1. C. Zener: "Mechanism of Armor Penetration, Third
   Partial Report", Report Number WAL 710/492-1.

instant of contact, not allowing it to exert any force
whatsoever. Armor plate without inertia if rigidly
supported at its boundaries would respond to a projectile
impact in precisely the same manner as if a force were
slowly applied. Such a plate would suffer extensive
bending. The inertia of actual armor plates, that is,
their resistance to acceleration, acts to a first approxi-
mation as a support during projectile impact.

The concept of plate support is not, however, without
ambiguity. Reference to Figure 1 shows that the effect
of a support depends not only upon its rigidity but also
upon the "tolerance" of the support and the projectile.
Thus in a die with a small tolerance, a plate which is
being punched suffers a nearly pure simple shear type of
deformation. On the other hand, when the clearance is
sufficiently large, the initial deformation of the plate
is nearly of the pure bending type. It is not clear,
without analysis, whether the support offered by the plate
inertia is of the close tolerance or of the large toler-
ance type, in other words, whether the plate's response
to a projectile impact is governed by shearing stresses
or by bending moments. Such questions are answered by
the analysis in the following sections.

In the study of the gross behavior of a plate sub-
jected to projectile impact, the boundary effects at the
edges of the plate introduce only complicating irrele-
vancies. These irrelevancies may be avoided by assuming

the duration of the impact to be so short that the projectile is no longer in contact with the plate when the elastic waves reach its boundary, or, what amounts to the same thing, by assuming the plate to extend laterally to infinity. In the following discussion, this second concept will be adopted.

A. Velocity of Region of Impact.

The elastic response of a laterally infinite plate to a concentrated normal force has already been examined in some detail.[1] It was found that when such a force is applied, the plate is displaced normally at the point of application of the force with a velocity which is proportional to the force. Thus

$$\text{Velocity} = \alpha \times \text{force} \tag{1}$$

The proportionality constant α is given by

$$\alpha = \left[\frac{3\rho(1-\sigma^2)}{E} \right]^{1/2} \Big/ 4\rho\, e^2 . \tag{2}$$

Here ρ is the density, σ Poisson's ratio, E Young's Modulus, and e the plate thickness.

The previous analysis is generalized in Appendix A to the case where the force is distributed over the face rather than concentrated at one point. Upon taking a

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1. C. Zener: "The Intrinsic Inelasticity of Large Plates" Physical Review, 59, 669 (1941).

Gaussian distribution, that is

$$\text{Pressure} = (\pi s^2)^{-1} \; e^{-r^2/s^2} \; f(t), \qquad (3)$$

it is found that the plate displacement at the center of the distribution is related to the total force $\underline{f}$ by

$$\text{displacement}(t) = \alpha \int_{-\infty}^{t} G(t-t') \; f(t') \; dt' . \qquad (4)$$

The function G is unity except for values of its argument comparable to, or less than, the time required for an elastic stress wave to traverse the distance $\underline{s}$, the distribution modulus. Its precise form is given by

$$G(t) = 1 - \frac{2}{\pi} \cdot \tan^{-1} p/t \qquad (5)$$

where

$$p = \left[3\rho (1-\sigma^2)/E \right]^{1/2} \cdot (s^2/2c) . \qquad (6)$$

In the special case of projectile impact, where the distribution modulus s is comparable to the projectile caliber, which in turn is comparable to the plate thickness, negligible error is introduced by taking the function G in Equation (4) as unity. In this case Equation (4) is equivalent to the simple equation (1).

In all but very exceptional cases, the velocity of the plate is only a small percentage of the incident velocity of the projectile. As an example, suppose a projectile strikes a matching plate (c/d = 1) at a velocity V of 2,000 f/s. The pressure exerted by the projectile

will be about ρV^2. Taking this pressure to act over
an area equal to the cross section of the projectile,
one finds that the plate velocity is at most only 3%
of the incident projectile velocity.

B. Bending Moment in Region of Impact.

A corollary of Equation (1) states that when the
force is removed, the displacement at the original center
of force distribution remains constant. The motion of
the plate after the force is removed is depicted sche-
matically in Figure 2.

From this figure it is evident that although the
velocity is zero at the original center of force distri-
bution, the curvature, and hence the tensile stress at
the back of the plate, approaches zero only asymptotically
after the removal of the force. This asymptotic behavior
suggests the following approach to the computation of the
bending moment, and hence of the tensile stresses, on the
back surface of the plate. Let the force f(t) be distri-
buted over the face of the plate as indicated in
Equation (3). If this force were non-vanishing only in
the time interval $\Delta t'$ at t', then the tensile stress
T(t) at any point on the back of the plate would be
proportional to $f(t')$ and to $\Delta t'$,

$$T(t) \sim f(t') \, \Delta t' \quad .$$

The constant of proportionality can be a function only

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of the time interval t–t', thus

$$T(t) = g(t-t') \ f(t') \ \Delta t'$$

If now no restrictions are put upon f for times prior
to t, the linearity of the elasticity equations insures
that the contributions to T(t) of f(t') during all
previous time elements $\Delta t'$ will be additive. Therefore

$$T(t) = \int_{-\infty}^{t} g(t-t') \ f(t') \ dt' \ . \tag{7}$$

The quantity g(t–t') may be called the influence
function. Its precise form will depend upon the precise
manner in which the force is distributed over the face
of the plate. It is evaluated in Appendix A for the
particular case of a Gaussian distribution, that is, for
a symmetrical distribution of the type given by Equation
(3). In the Appendix it is shown that the tensile stress
on the back surface symmetrically behind the force is
given by Equation (7) with the following influence function:

$$g(\tau) = (3/4\pi) \ (1 + \sigma) \ e^{-2} \left(\frac{\tau}{\tau^2 + q^2} \right) , \tag{8}$$

where e is the plate thickness, and where

$$q = (3^{1/2}/2) \ (s^2/e) \ \left[(1-\sigma) \rho/E \right]^{1/2} . \tag{9}$$

The quantity q is effectively the time required for an
elastic wave to travel the distance s^2/e.

As an example of the above analysis, suppose a force

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with the Gaussian distribution of Equation (3) is suddenly
applied at t = o, and maintained at the constant value f.
Then the integration of Equation (7) leads to

$$T(t) = (f/2\pi e^2) \cdot \ln(1 + t^2/q^2) , \qquad (10)$$

in which equation σ has been replaced by 1/3. A plot of
T(t) vs. t is given in Figure 3. When t is large compared
with q, this equation reduces to

$$T(t) = (f/\pi e^2) \cdot \ln(t/q), \quad t \gg q . \qquad (11)$$

The increase of the tensile stress T with time, as
given by Equation (10) or (11), is the basis for the type
of projectile used in shock testing plates. In these
tests it is desired to subject the back of the plate to
tensile stresses sufficiently large to cause plastic de-
formation in tension at the back of the plate, at the
same time keeping the shearing stresses sufficiently low
to prevent a punching from being pushed out. The ap-
propriate stress pattern is obtained by the use of a soft
projectile which mushrooms upon the plate, thereby pro-
longing the time of impact.

II DETAILED BEHAVIOR OF PLATE.

In the first part of this report a study was made
of the response of the plate as a whole to an applied
force. The information so obtained will now be used to
simplify the study of the behavior of the plate material

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immediately surrounding a projectile. This simpli-
fication is accomplished by replacing the plate material
outside of a certain cylindrical surface by a rigid
support, as indicated in Figure 4, with a diameter greater
than, but comparable to, the caliber of the projectile.
The boundary conditions at the support are so chosen
that the support, and hence the inclosed plate, move
normally with the velocity, given by Equation (1), with
which the plate actually moves at the center of the
region of impact, and so that the moment with which the
support acts on the inclosed plate is equal to the
moment, whose tensile stresses are given by Equation (7),
of the actual plate at the center of the region of
impact.

A. Low Velocity Case of Quasi-Static Motion.

When the above mentioned concept of plate support
is adopted, the manner in which the plate material is
deformed at any instant may be regarded as independent
of the velocity, provided the velocity is not too high,
and dependent only upon the position and direction of
motion of the projectile. Thus an increase of velocity
by a given factor would increase the rate of strain in
every element of the plate by the same factor. This
invariancy of type of deformation with projectile ve-
locity does not necessitate an invariancy of stress
distribution with velocity. In fact, due to the inertia

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of the plate material, the stress pattern gradually changes with projectile velocity. The stress pattern may be regarded as that arising from the forces due to the projectile, the support, and a certain distribution of body forces. The body force acting upon each element of volume is considered to be the mass of the element, multiplied by the negative of its acceleration.

The effect of the inertia of the plate material inside the assumed support upon the force with which the plate reacts on the projectile may be readily computed without explicitly computing its effect on the stress pattern throughout the plate. It is only necessary to compute the total kinetic energy of the plate, K.E., as a function of the velocity and position of the projectile. The partial derivative of this kinetic energy with respect to a spatial coordinate of the projectile gives the component of force acting along the negative axis of this coordinate.

The above mentioned method of computing the effect of the plate material's inertia upon the force with which it resists penetration may be most simply illustrated by the case of normal incidence. In this case the force is normal to the plate, and is directed along the axis of the projectile. Let s be the distance which the projectile has penetrated into the plates. Then that portion of the plate's resistance which is due to its inertia is given

by the following equation:

$$\text{Inertial force} = \frac{\partial}{\partial s} (\text{K.E.}) . \qquad (12)$$

Special significance is to be attached to the notation
of the partial derivative. The kinetic energy is a
function not only of the positional variable s, but also
of the velocity V. In the differentiation of Equation
(12), cognizance is not to be taken of the fact that V
varies with s, but V is to be held constant. The kinetic
energy will be proportional to ρV^2, where ρ is the
density of the plate material. Thus

$$\text{K.E.} = A(s) \rho V^2 . \qquad (13)$$

The constant of proportionality A has the dimension of
volume.

The plate's kinetic energy, and the inertial force
associated therewith, are plotted schematically in
Figure 5 for projectiles with two types of ogives. For
all types of ogives the inertial force has one common
characteristic, the area beneath the inertial force vs.
penetration distance is identically zero. Thus this force
opposes the projectile as it enters the plate, but aids
the projectile as it leaves. The kinetic energy function
changes in a characteristic manner as the ogive of the
projectile is made blunter; the maximum value is raised,
and the initial rise is steeper. These changes in the

kinetic energy function are reflected in the inertial force function by a sharp rise in the initial part.

The initial high inertial force may, when added to the force arising from the resistance of the plate's material to plastic deformation, be so high as to give rise to projectile shatter. In attack against homogeneous armor, it is the function of A.P. caps to lower the initial inertial pressure. This lowering is effected by a smoothing of the initial rise in kinetic energy, as illustrated in Figure 6, which smoothing results from the effective lengthening of the time required for the ogive to dig completely into the plate. In contradiction to the above interpretation of the action of A.P. caps, it is commonly stated that the function of an A.P. cap is to lend lateral support to the projectile. In order to demonstrate the correctness of the first point of view, namely the smoothing out of the initial inertial force, projectiles were made whose ogives were flattened cones. The shatter characteristics of these projectiles were examined for the two cases of bare ogives and of ogives protected by caps on the flat faces. (The caps were disks of copper 1/32" thick). A comparison of their behavior is presented in Figure 7. Since the A.P. caps on these projectiles did smooth out the initial inertial force, while they could not provide lateral support, it is apparent that, at least in these cases, the action of

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A.P. caps is related to the initial inertial force.

In the case of oblique impact the inertial force of the plate reacting upon the projectile is no longer directed along the projectile's axis. Any component of this force may be computed by an equation analogous to Equation 12. Thus suppose one wishes to know the component of the inertial force normal to the surface of the plate. Letting x denote the position of the center of gravity of the projectile from the plate, one then has

$$\text{(Inertial force)}_{\text{normal component}} = \frac{\partial}{\partial x} \text{(K.E.)}$$

The interpretation of the differentiation in this equation is given in Figure 8. Thus consider a fictitious projectile which has the same orientation and the same velocity as the actual projectile, but which is displaced with respect to the plate by the distance Δx. Let $\Delta K.E.$ be the increase of the kinetic energy of the plate in the case of the fictitious projectile over that caused by the actual projectile. Then $\partial K.E./\partial x$ denotes the limit of the ratio $\Delta K.E./\Delta x$ as Δx approaches zero.

B. High Velocity Case of Stress Waves.

In the previous section it was assumed that the strain distribution in the plate material immediately surrounding a penetrating projectile was independent of the projectile's velocity. This assumption enabled us, at least in principle, to compute the changes in stress distribution arising

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from the inertia of the plate material. When this change in stress distribution becomes so great that the elastic strain associated therewith makes an appreciable change in the deformation, the original simplifying assumption of quasi-static deformation is no longer valid. An accurate description of the plate's behavior can then be obtained only by an analysis of the fundamental equations of motion of the plate material.

Some very important qualitative conclusions may be drawn merely from a consideration of the general nature of the solutions to the fundamental equations of motion of the plate material. The following properties of these solutions are of particular importance to the present discussion.

1. Stress and deformation are propagated with a finite velocity.

2. When a stress wave reaches a free surface, a stress wave of opposite sign is reflected (e.g., a compressive wave is reflected as a tensile wave).

The first property makes it possible for an impulsive compressive wave to exist in the plate. Thus suppose that, due to the plate material's inertia, the projectile exerts a very large force upon the plate for a very short interval of time Δt. The product of Δt and of the velocity of stress propagation, c, gives the length, Δx, of the compressive stress impulse,

$$\Delta x = c \, \Delta t \; .$$

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If the time interval Δt is so short that Δx is less than the plate thickness, the projectile sets up an impulsive compressive wave running across the plate. The time interval Δt is shorter the higher the striking velocity and the blunter the ogive.

The second property is responsible for the transformation of a compressive impulsive wave travelling towards the back of the plate into a tensile impulsive wave travelling from the back to the face of the plate. If the fracture stress of the plate is sufficiently low across a plane parallel to the plate, this tensile wave will result in the throwing off of a lamination. Examples are shown in Figure 9. In this figure is shown the change in the type of plate failure as the velocity of the projectile was raised. The ogive of the projectile was sufficiently blunt so that, at the lower velocities, a punching was found. At higher velocities, a distinct circular lamination was thrown off the back of the plate.

The force with which the projectile acts upon the plate is exactly equal to the force with which the plate acts upon the projectile. It is therefore to be expected that an impulsive compressional wave will run down the projectile in an opposite direction to that of the compressional wave in the plate. This compressional wave is reflected at the base of the projectile as a tensile wave. As this tensile wave travels back into the ogive, the

intensity of the stress increases as the area of the wave front decreases. If the intensity of the tensile stress reaches the fracture stress of the projectile material, fracture will result. An example of such a fracture is shown as Figure 10.

The assumptions of quasi-static deformation break down completely when the projectile presents a flat or curved surface to the plate. In such cases the assumption of quasi-static deformation would imply that the plate material acquired a finite kinetic energy in an infinitesimal interval of time at the moment of impact. In order to estimate the magnitude of the pressure which actually exists between the projectile and the plate in such cases, a detailed analysis is necessary. The results of such an analysis are discussed below.

Just after the projectile has made contact with the plate, the plate material immediately in front of the area of contact moves normally to the plate, lateral motion being prevented by the inertia of the plate material. At the first instant of contact one is therefore justified in treating the impact as a one dimensional problem. It may readily be seen that for this case the pressure is given by the product (density of material) x (velocity of stress propagation) x (velocity of interface of projectile and plate). The velocity of a wave in which transverse motion is prevented, and in which the resistance to shear

deformation may be neglected, is $(K/\rho)^{1/2}$, where K is the bulk modulus. This velocity will be denoted by c. This velocity is very nearly equal to the velocity of propagation of an elastic wave along a bar, namely 16,000 f/s. The initial velocity of the interface of the projectile and plate must be taken as one half of the projectile's velocity V, since at the first instant of contact a wave of equal magnitude but of opposite direction is set up in the projectile. Thus

$$\text{initial inertial pressure} = (1/2) \, \rho \, cV \, . \quad (14)$$

This pressure is plotted in Figure 11.

If the wave motion remained strictly one dimensional, the pressure would remain constant at the value given by Equation (14) until the return of a wave reflected from a boundary. Actually, the spreading of the wave into the transverse directions results in a rapid decrease in the pressure. This difference in the behavior of one and of three dimensional waves is discussed analytically in Appendix B. It is there found that in the three dimensional case the initial inertial pressure decreases exponentially from its peak value. Thus,

$$\text{initial inertial pressure} = 1/2 \, \rho cV \, e^{-t/\tau}.$$

The time constant τ is approximately equal to the ratio of the diameter of the contact region to the velocity with which the stress wave is propagated.

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APPENDIX A

In this appendix the theory is developed for the influence functions, G of Equation (4) and g of Equation (7), which relate the displacement of the plate, and the maximum tensile stress on the back surface, respectively, of a laterally infinite plate to the force acting upon the face. An explicit expression is obtained for the particular case of a Gaussian distribution, as defined by Equation (3).

The analysis will follow closely that given by the author[1] for the relation between a concentrated force acting upon a laterally infinite plate and the velocity produced thereby. In this analysis we start from the usual approximate theory of thin plates.[2] In this theory it is assumed that the radius of curvature of the plate is everywhere large compared with the plate thickness, and that the angle between the plate and the original plane is everywhere small. Upon taking the x,y plane to be parallel to the original plane of the plate, one has the following approximate equation for the transverse displacement $U(x,y,t)$ of the plate:

$$(D\nabla^4 + m\partial^2/\partial t^2)\, U = Z \ . \qquad (a\text{-}1)$$

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1. C. Zener: "The Intrinsic Inelasticity of Large Plates", Physical Review 59, 669 (1941).
2. A. E. H. Love: "Mathematical Theory of Elasticity", (Cambridge, 1927) Fourth Edition, p. 487.

In this equation, m is the mass of a unit area of plate, and $Z(x,y,t)$ is the surface density of the normal force. The rigidity modulus D is defined by

$$D = (1/12) e^3 E', \quad E' = E/(1-\sigma^2) ,$$

where e is the plate thickness, E is Young's Modulus and σ is Poisson's ratio. The operator ∇ is defined by the following equation:

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 .$$

In Equation (a-1), $Z(x,y,t)$ is to be regarded as a known function, $U(x,y,t)$ as an unknown function. The formal solution for U will be obtained in terms of the eigenfunctions of the auxiliary equation:

$$\left[(D/m)^{1/2} \nabla^2 + W \right] U(x,y) = 0 , \qquad (a-2)$$

and of the boundary conditions at the edge of the plate. The eigenvalues and normalized eigenfunctions of these equations will be noted by W_n and U_n, respectively, where the suffix $\underline{n}$ refers to an ensemble of two characteristic numbers. The formal solution will therefore be written as

$$U(x,y,t) = \Sigma_n Cn(t) U_n(x,y) \qquad (a-3)$$

The general coefficient C_n in the above summation will be obtained by substituting this equation for U into Equation (a-1), multiplying by U_n, and integrating over

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the entire surface of the plate. Upon using the ortho-
gonality property of the eigenfunctions one obtains

$$(d^2/dt^2 + W_n{}^2) \, C_n(t) = m^{-1} \, Z_n(t) \, , \qquad \text{(a-4)}$$

where

$$Z_n(t) = \int U_n \quad Z(x,y,t) \quad dA \, , \qquad \text{(a-5)}$$

dA denoting an element of area of the plate.

Corresponding to the condition that the displacement
U, as well as the time derivative of the displacement will
be taken to be everywhere zero at $t = 0$, that solution of
Equation (a-4) will be obtained which satisfies the con-
ditions

$$\left. \begin{array}{l} C = 0 \\ dC/dt = 0 \end{array} \right\} \text{ at } t = 0 \qquad \text{(a-6)}$$

That solution of Equation (a-4) which satisfies the boundary
condition (a-6) is

$$C_n = (m \, W_n)^{-1} \int_0^t \, Z_n(t') \, \sin W_n(t-t') \, dt' \, . \quad \text{(a-7)}$$

The displacement U is now obtained by substituting C_n frcm
Equation (a-7) into Equation (a-3). The result is

$$U(x,y,t) = \Sigma_n (m \, W_n)^{-1} \, U_n(x,y) \cdot$$

$$\int_0^t \, Z_n(t') \, \sin W_n(t-t') \, dt' \qquad \text{(a-8)}$$

In computing the stress or the back surface of the
plate, we need the quantity $\nabla^2 U$. Operating on both sides

of Equation (a-8) with ∇^2, and using Equation (a-2),
we obtain

$$\nabla^2 U = -(mD)^{-1/2} \Sigma_n \, U_n(x,y) \cdot \int_0^t Z_n(t') \, \sin W_n(t-t') \, dt' ,$$

$$(a-8')$$

Before proceeding further with either Equation (a-8)
or Equation (a-8'), it is necessary to define the shape
of the plate and to specify the boundary conditions to
which the plate is subjected. The plate will be taken
as a square lying in the region

$$0 < x < L$$
$$0 < y < L .$$

The boundary conditions will be so chosen that the plate
is free to pivot along its edges. The normalized eigen-
functions of the differential equation (a-2) are then

$$U_{(j,k)} = (2/L) \, \sin(\pi j x/L) \, \sin(\pi k y/L) , \quad j,k = 1,2,\cdots$$

The corresponding eigenwert is

$$W_{(j,k)} = (D/m)^{1/2} \, (\pi/L)^2 \, (j^2 + k^2). \quad (a-9)$$

The surface density of force $Z(x,y,t)$ will be taken
as given by p of Equation (3), with the center of the
distribution at the center of the plate, i.e., at
$x = L/2$, $y = L/2$. At the center, $U_{(j,k)}$ is different from
zero only when both j and k are odd. When j and k are
both odd one obtains, using Equations (a-5) and (a-9),

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that at center of plate

$$U_{(j,k)} \, Z_{(j,k)} = 4L^{-2} \, f(t) \, \exp\left\{ -(m/D)^{1/2} \, (s^2/4) \, W_{(j,k)} \right\}$$
(a-10)

This equation for $U_n \, Z_n$ is now substituted into Equations (a-8) and (a-8'), and the summation Σ_n performed over all characteristic numbers j, k. The number of ensembles (j,k) for which both j and k are even, and for which $W_{(j,k)}$ lies in the range dW at W, may be seen from Equation (a-2) to be given by

$$\left[L^2 \, (m/D)^{1/2}/16\pi \right] \, dW .$$

and therefore the summation $\Sigma_n \cdots$ may be replaced by an integral as follows

$$\Sigma_n \cdots = \left[L^2 \, (m/D)^{1/2}/16\pi \right] \int_0^\infty dW \cdots \quad \text{(a-11)}$$

Upon applying Equations (a-10) and (a-11) to Equation (a-8), one obtains Equations (4)-(6) of the text. In the integration with respect to t', one makes use of the formula

$$\int_0^\infty e^{-ax} \, x^{-1} \, \sin x \, dx = \pi/2 - \tan^{-1} a ,$$

which may be obtained by integration of the following standard integration formula with respect to $\underline{b}$ over the range $\underline{a}$ to ∞:

$$\int_0^\infty e^{-bx} \, \sin x \, dx = 1/(b^2 + 1) .$$

Applying Equations (a-10) and (a-11) to Equation

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(a-8'), one obtains at the center of plate

$$\nabla^2 U = -(4\pi D)^{-1} \int_0^t f(t') \cdot (t-t') \cdot \left[(t-t')^2 + q^2 \right]^{-1} dt' ,$$

where

$$q = (m/D)^{1/2} s^2/4 .$$

Upon utilizing the definitions of m and of D, this equation for q is seen to be equivalent to Equation (9).

It remains to pass from $\nabla^2 U$ to the stresses in the back of the plate. If M_1 and M_2 refer to the bending moments, per unit length, along two orthogonal directions, then[1]

$$M_1 + M_2 = -D(1 + \sigma) \nabla^2 U ,$$

where σ is Poisson's ratio. In the center of the pressure distribution M_1 and M_2 are equal. Denoting their value by M, we therefore obtain

$$M = (1 + \sigma) (8\pi)^{-1} \int_0^t f(t') (t-t') \left[(t-t')^2 + q^2 \right]^{-1} dt' . \tag{a-11}$$

In the elastic range the moment M and the tensile stress T at the surface are related by

$$M = (e^2/6) T . \tag{a-12}$$

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1. R. V. Southwell: Theory of Elasticity (Oxford, 1941), p. 230.

Upon substitution of Equation (a-12) into Equation (a-11), one obtains Equation (7) with the influence function g given by Equation (8).

APPENDIX B

In this appendix an analysis will be given of the difference in pressures necessary suddenly to start a boundary moving at a constant velocity in the one dimensional case and in the three dimensional case.

In both cases the medium will be regarded as possessing no resistance to shear deformation, i.e., the medium will be considered as a perfect fluid. The bulk modulus of this fluid will be denoted by K. In the one dimensional case lateral expansion will be prevented by lateral constraints.

The equations of motion and the continuity equations have the following solutions.

1. One dimensional case.

The velocity V and pressure P are given in terms of a velocity potential Φ through the equations

$$V = -\ \partial\Phi/\partial x, \qquad\qquad (b\text{-}1)$$

$$P = K\rho\ \partial\Phi/\partial t . \qquad\qquad (b\text{-}2)$$

The velocity potential depends upon x and t as follows,

$$\Phi = f(x-ct) \qquad\qquad (b\text{-}3)$$

where f is an arbitrary function of its argument, and where $c^2 = K/\rho$. Substitution of Equation (b-3) into Equations (b-1) and (b-2) gives

$$P = \rho cV \qquad\qquad (b\text{-}4)$$

This equation is of general validity, independent of the variation of V with time.

2. Three dimensional case.

In this case the medium is assumed to extend to infinity in every direction, and to contain a spherical cavity. The walls of this cavity are further assumed to suddenly start to move radially outwards with a constant velocity. The pressure on the walls will be computed as a function of time.

The velocity and pressure are given by equations analogous to Equations (b-1) and (b-2), namely[1]

$$V = - \partial \Phi / \partial r \qquad (b-5)$$

$$P = K \rho \, \partial \Phi / \partial t . \qquad (b-6)$$

The velocity potential is given by

$$\Phi = \frac{f(r-ct)}{r} , \qquad (b-7)$$

where, as before, f is an arbitrary function of its argument.

Substitution of Equation (b-7) into Equation (b-5) leads to

$$\partial f / \partial r - r^{-1} f = -r V.$$

Since the argument of f is r-ct, the above may be re-written as

$$\partial f / \partial t + (c/r) f = c \, r \, V .$$

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1. H. Lamb, Hydrodynamics (Cambridge Press, 1932), Sixth edition, p. 489.

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We now suppose that V is some known function of t at $r = a$. Then at this value of r,

$$\partial f / \partial t + (c/a) f = ca\, V_a \quad .$$

We shall seek the solution of this equation which vanishes at some definite time, say at $t = 0$. This solution is

$$f = a^2 V_a\, (1 - e^{-(c/a)t}), \quad r = a \quad .$$

Substitution of this equation back into Equation (b-7) and then into Equation (b-6), leads to
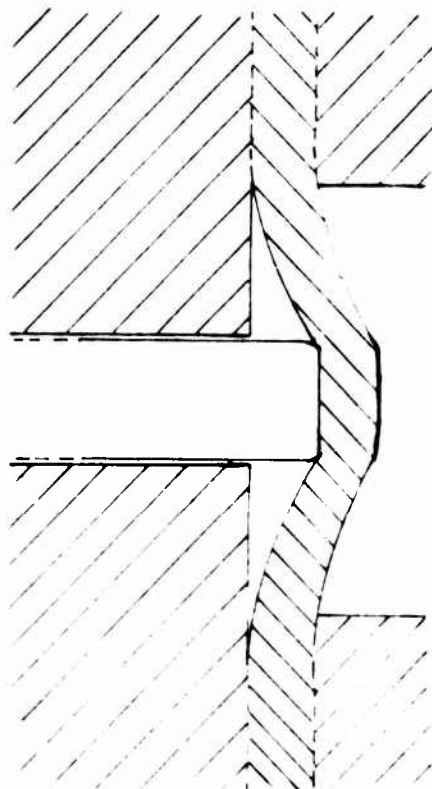
$$p = \rho c\, V_a\, e^{-(c/a)t}, \quad r = a \quad . \tag{b-8}$$

Upon comparing Equations (b-4) and (b-8), we see that the pressure starts off with the same values when the boundaries begin to move. While the pressure remains constant in the one dimensional case, it decreases exponentially in the three dimensional case.

It may be noted that Equation (b-8) is strictly the solution to the case where the velocity remains constant at $r = a$, rather than at the moving boundary. This difference in problems introduces no essential change in the solution.
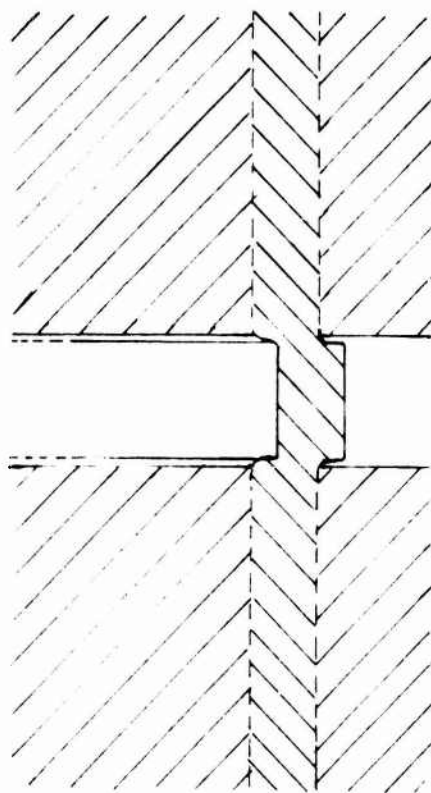
FIGURE I

ILLUSTRATION OF INFLUENCE OF TOLERANCE OF
SUPPORT UPON TYPE OF DEFORMATION

b. SMALL TOLERANCE

DEFORMATION IS PURE SIMPLE SHEAR

a. LARGE TOLERANCE

DEFORMATION IS PURE BENDING.

FIGURE 2

SCHEMATIC DIAGRAM OF MOTION OF PLATE
AFTER REMOVAL OF FORCE

PLATE AT SUCCESSIVE TIMES
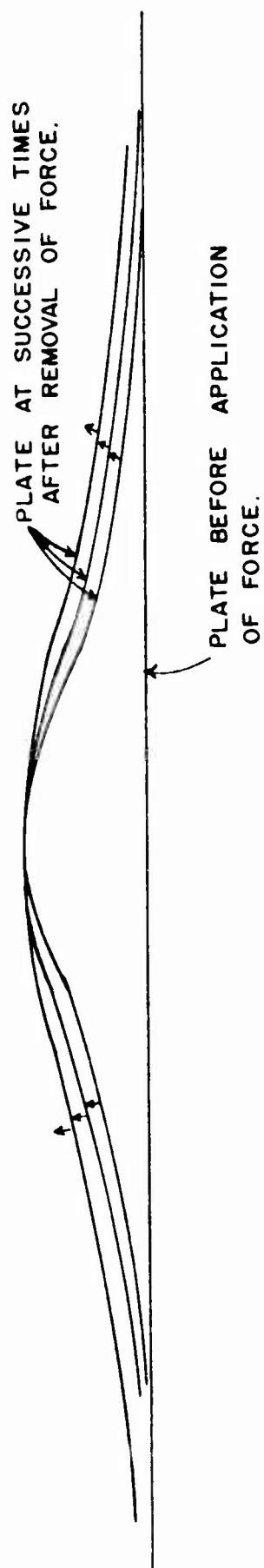AFTER REMOVAL OF FORCE.

PLATE BEFORE APPLICATION
OF FORCE.

FIGURE 3

GROWTH OF TENSILE STRESS ON BACK OF PLATE WITH TIME
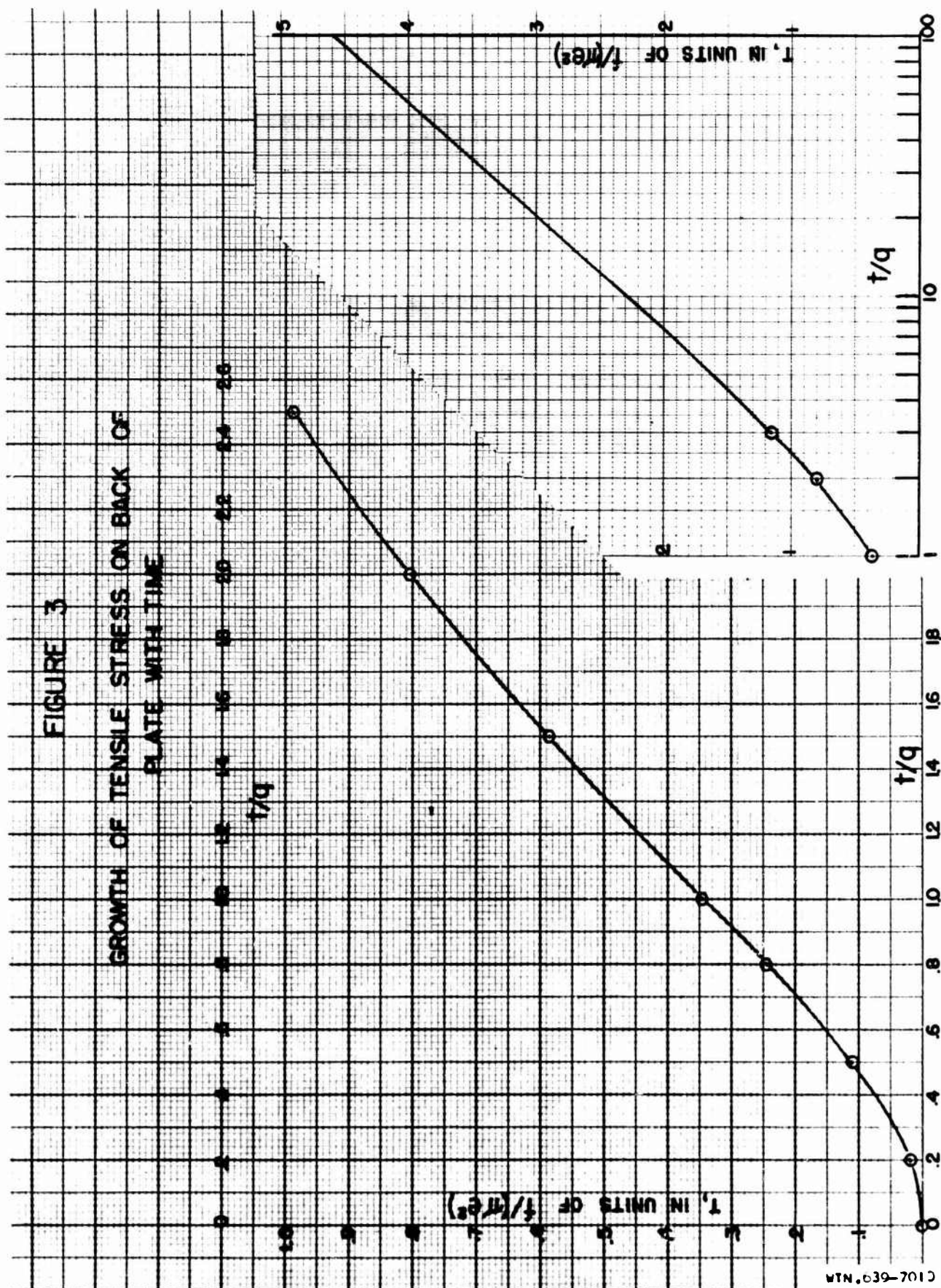
FIGURE 4

ILLUSTRATION OF METHOD OF SIMPLIFYING THE

MECHANICS OF ARMOR PENETRATION

a. ACTUAL SYSTEM

b. SIMPLIFIED SYSTEM
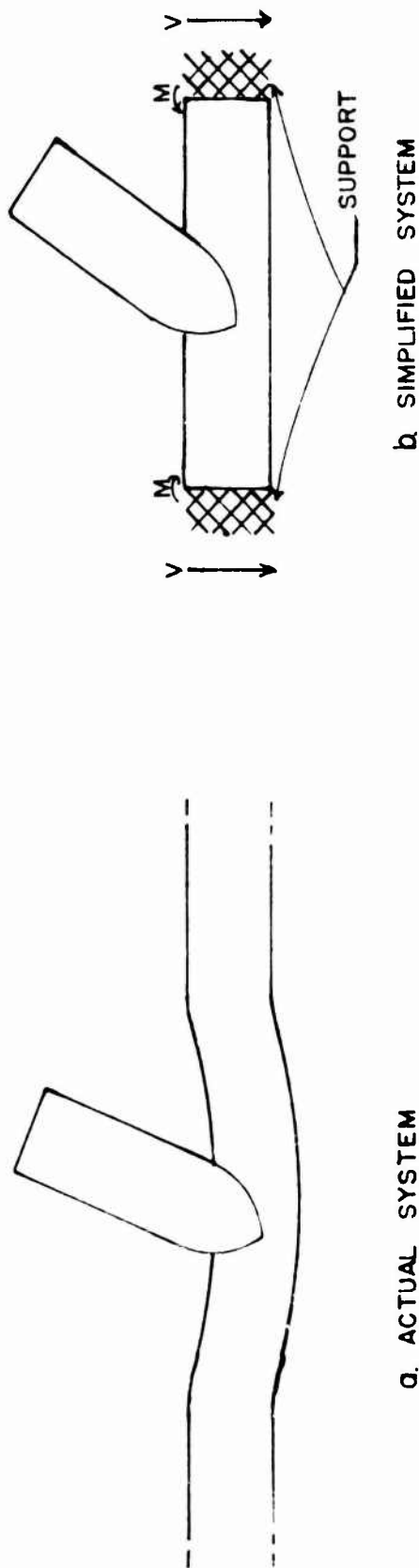
VELOCITY, V, OF SUPPORT, AND MOMENT, M,
EXERTED BY SUPPORT, IS EQUAL TO VELOCITY
AND MOMENT, RESPECTIVELY, OF ACTUAL PLATE
IN IMMEDIATE VICINITY OF PROJECTILE.

FIGURE 5

EXAMPLES OF INERTIAL FORCE (SCHEMATIC)
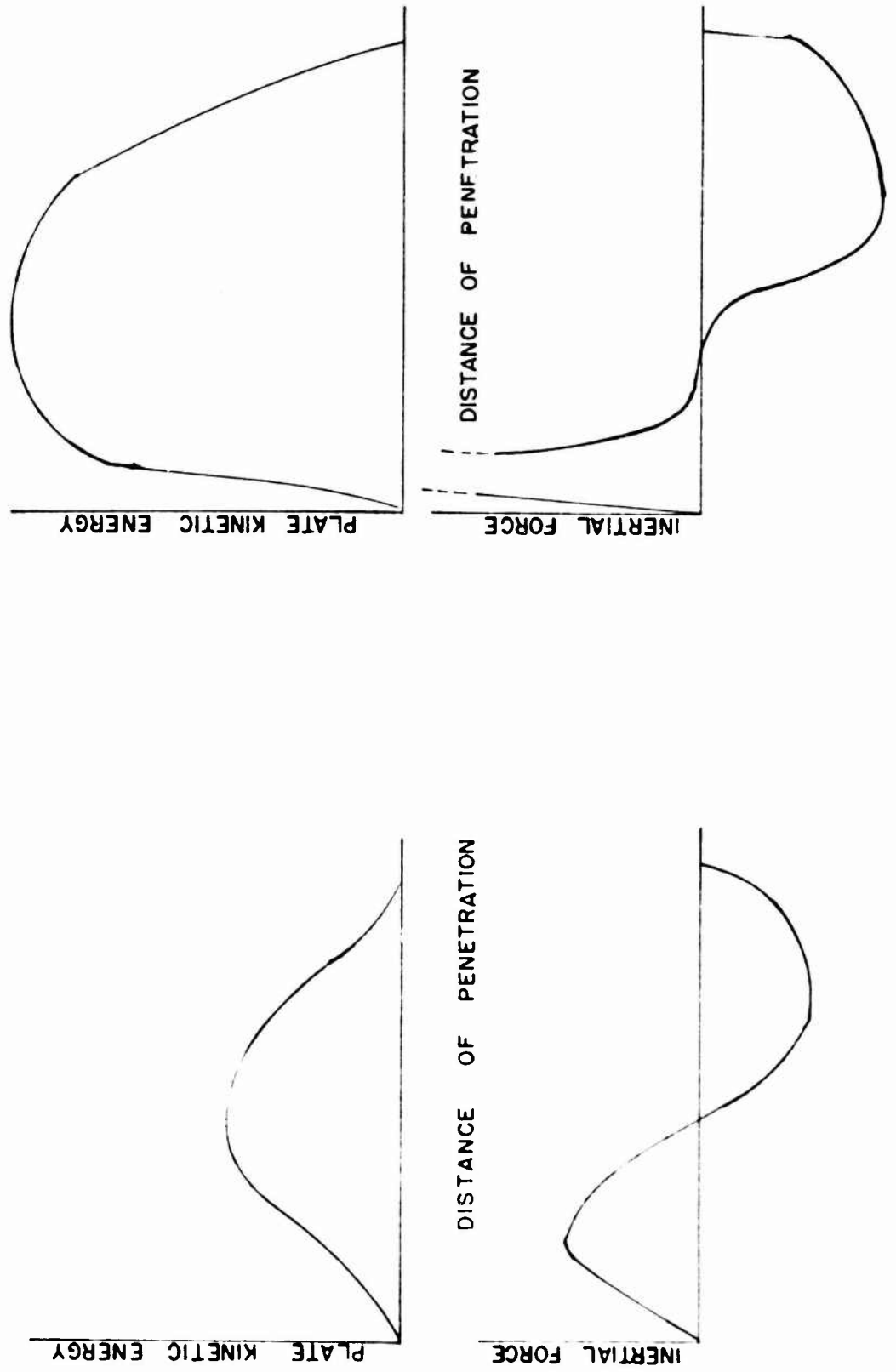
a. CASE OF LONG OGIVE

b. CASE OF SHORT OGIVE

FIGURE 6

ILLUSTRATION OF HOW AN A.P. CAP REDUCES
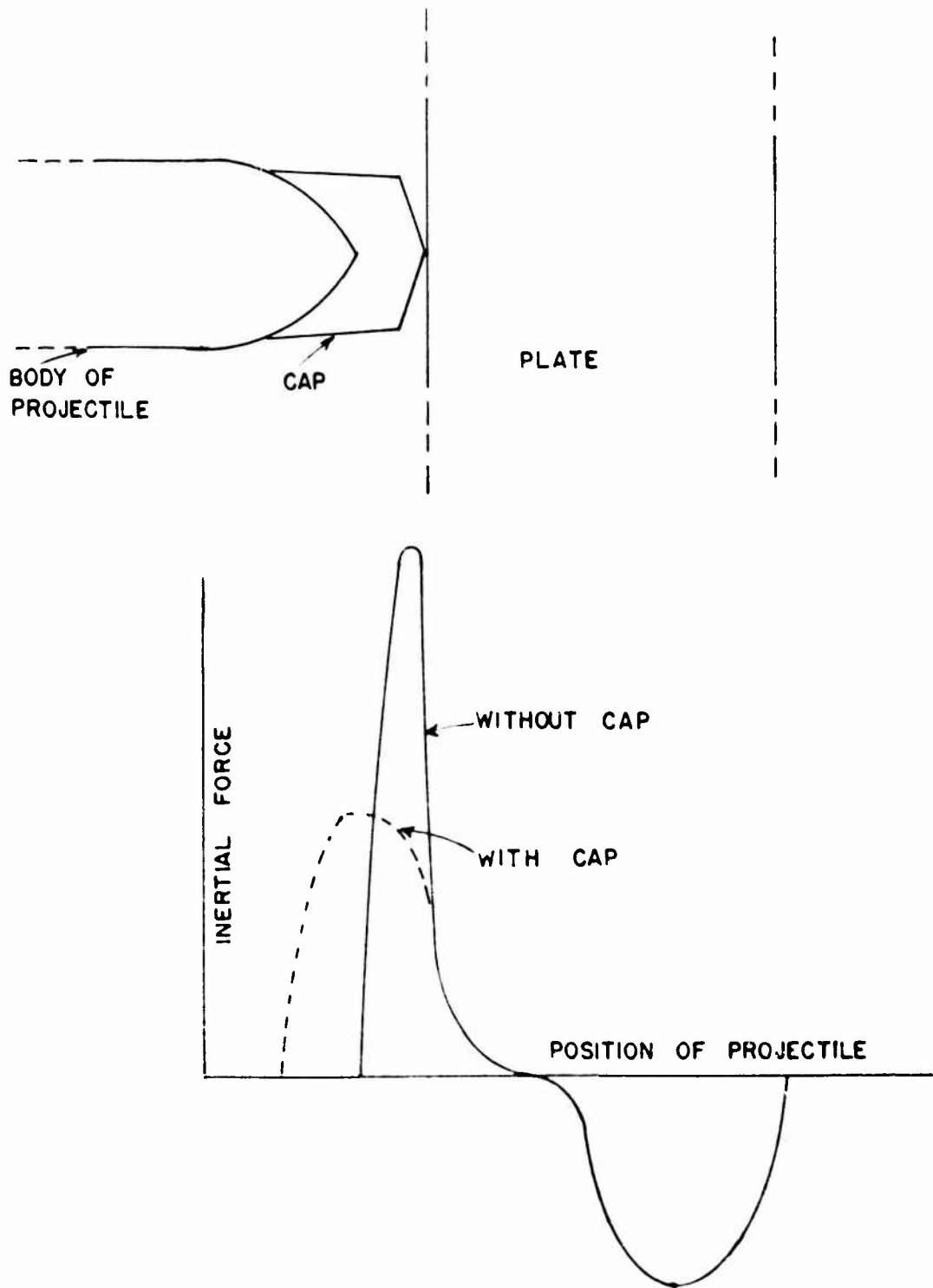INITIAL INERTIAL FORCE

FIGURE 7

ILLUSTRATION OF BUFFER ACTION
OF CAP

CAPPED   UNCAPPED

UNFIRED PROJECTILES

CAPPED   UNCAPPED

RECOVERED PROJECTILES

(INCIDENT VELOCITY: 2000 f/s
  PLATE: 0.25", 321 BHN)

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FIGURE 8

ILLUSTRATION OF METHOD OF COMPUTING THE NORMAL COMPONENT OF INERTIAL FORCE

NORMAL COMPONENT OF
INERTIAL FORCE

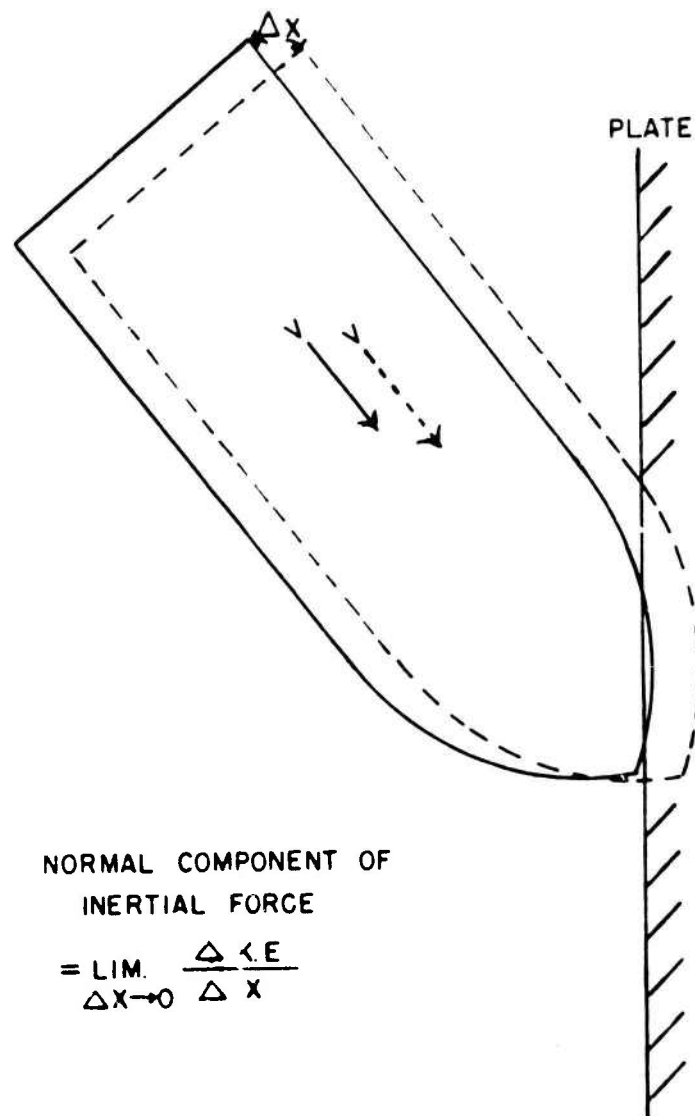$$= \lim_{\Delta X \to 0} \frac{\Delta \, K.E}{\Delta \, X}$$

FIGURE 9

ILLUSTRATION OF EFFECT OF REFLECTION OF COMPRESSIVE WAVE AT BACK OF PLATE

V = 1255

PURE SHEAR PUNCH
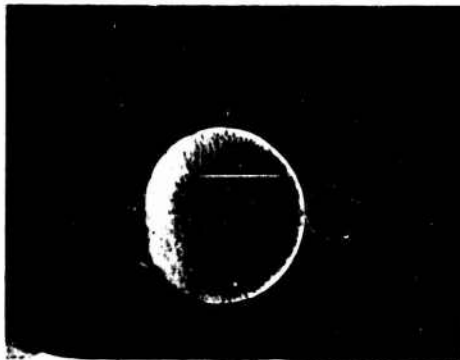
V = 1620

BACK SPALL

(PLATE:— .30" THICK, 321 BHN;
PROJECTILE:— 62 ROCKWELL "C", NORMAL INCIDENCE).

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FIGURE 10

ILLUSTRATION OF EFFECT OF REFLECTION
OF COMPRESSIVE WAVE AT BACK OF PROJECTILE

(PLATE THICKNESS = 2"; PROJECTILE 37MM BODY WITH POINT ON OGIVE BLUNTED; V= 2350 f/s).

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FIGURE II

INITIAL INERTIAL PRESSURE IN CASE PROJECTILE PRESENTS A TANGENT SURFACE TO PLATE AT IMPACT